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Controlling one-dimensional unimodal population maps by harvesting at a constant rate

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Controlling chaos and periodic oscillations in dynamical systems is a well known problem. It has been studied by several types of algorithms, some of which were only demonstrated numerically with specific examples. We study here a class of one-dimensional discrete maps of the form $x_{n+1}=f(x_n)$, where $f(x)$ is a unimodal function satisfying a few natural conditions as a model for the dynamics of a single species population. For managing the population we seek to suppress any possible chaotic or periodic behavior that may emerge. The paper proposes a simple, rigorously proved algorithm for controlling unimodal maps, implemented by harvesting the population at a constant rate. This forces the orbits of the map to converge to a limit that we can compute *a priori*. The result holds for any initial conditions within an interval that we specify. Our control algorithm is easy to implement, requires no updated information on the population and no changes in the parameters of the system, which, in general, are fixed by the properties of the population. It can therefore be useful for exploiting a population while maintaining it at a fixed density. [S1063-651X(98)08103-3]

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I. INTRODUCTION: BACKGROUND AND MOTIVATION

One-dimensional maps of the form

$$x_{n+1}=f(x_n) \tag{1}$$

have been used for modeling the dynamics of single species populations with nonoverlapping generations (e.g., [1–3]). Comparisons of model predictions to field data were discussed (e.g., [4–7]). One-dimensional maps were also used as theoretical examples for producing chaotic orbits with very simple dynamics (e.g., [2,8]), and some theory of discrete maps can be found in [9] and [10].

We are interested here in a special class of "single humped" maps [11] where f: $I\rightarrow I$ is a C^2 unimodal function, on the interval $I\subseteq[0,\infty)$. We also impose here a few natural conditions on $f(x)$, as the map supposedly describes the growth of a population and is also assumed to display rich dynamics (needed to be suppressed). We thus assume that $f(0)=0$, $f'(0)>1$, and that there exist two fixed points satisfying $f(x)=x$. The first such point is $x=0$ and it is an *unstable* fixed point of the map (1). The second occurs at some $x_Q>0$, past the peak of $f(x)$. We further assume that $f'(x_Q)<-1$, to ensure that x_Q is also an unstable fixed point of the map (1). Rich dynamics, such as chaotic oscillations that we may want to suppress, would not occur otherwise. Finally, we assume that $f(x)<x$ for $x>x_Q$, and that $f''(x)<0$ for $0\leqslant x\leqslant B$ for some $B>x_Q$. Typically, the maps used in the literature depend on one or more parameters that "tune" their dynamics. They satisfy the above conditions at least in some parameter range. Some examples are the classical population models such as the logistic map where $f(x)=rx(1-x)$, the exponential map where $f(x)=xe^{r(1-x)}$, the maps $f(x)=rx/[1+(ax)^b]$ [12], $f(x)=rx/(1+ax)^b$ [4,5], $f(x)=\lambda x(1+x+c/x)^{-\beta}$ [3], and others. Here r,a,b,c,λ,β are positive parameters used for tuning the respective models (ideally fit them to data). All of these maps exhibit periodic and chaotic oscillations in some parameter range.

It is well known that even simple maps can display chaotic dynamics, and the problem of suppressing such chaos has therefore drawn some research effort. Several different chaos suppressing algorithms were applied to discrete systems, systems of ordinary differential equations, and experimental setups. One approach stabilizes chaos by changing the system parameters (e.g., [13–15]). Other techniques apply continuous feedback [16], weak external periodic forces [17], small perturbations [18], or add noise to the system (see [19] where several methods were used for controlling chaos in the BVP oscillator).

Often it is not easy to derive rigorous conclusions in this context, particularly in discrete systems. Thus, some of the above, as well as many other related publications, propose methods that are merely tested with specific examples and supported only by numerical evidence. Consequently, some difficulties may arise. Consider, for example, Güémez and Matias [20] who studied one-dimensional discrete maps and proposed to control chaos by occasional application of proportional feedback, i.e., changing x_n into $x'_n=(1+\gamma)x_n$ every Δn iterations. To support their proposed algorithm, they demonstrated it with the logistic and the exponential maps for some specific choice of parameters, showing several numerically computed orbits. It was explained that a large number of iterations was calculated to assure the periodicity of the resulting orbits, which did not set in immediately. Although this algorithm was reported to depend strongly on γ and Δn, no method was provided for obtaining appropriate values (or range) of these parameters, or even the sign of γ. It is therefore not clear that this algorithm would be successful with other values of γ and Δn, with another choice of the tuning parameter r, with a different choice of initial conditions, or with other maps. In fact, it is not even proved that this algorithm did work for the two tested maps because observations of a finite number of iterations cannot assure periodicity or convergence. The two maps mentioned in [20]

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were partly motivated as a description of population growth. However, it is not clear that the proposed algorithm is practical in this context, since one needs to know x_n when the perturbation is applied, and then add (if $\gamma>0$) a proportional number of individuals to the system.

In this paper we study a simple algorithm for suppressing chaos or any periodic oscillations in a unimodal maps. It is implemented by harvesting the population at a constant rate, that is by modifying the map (1) to

$$x_{n+1}=f(x_n)-K \tag{2}$$

with some constant $K>0$. This constant feedback method has a practical advantage in the following sense: It requires no updated information on the state of the system because (unlike adaptive chaos control algorithms used in cases where the map itself is unknown) it does not use the value of x_n. Unlike other methods, it does not change any of the system parameters, which, in general, may be fixed by the properties of the population. Finally, this control algorithm is easy to implement in practice, and more important, $K>0$ implies that control is achieved by *harvesting* the population. To the extent that such one-dimensional maps indeed capture the essential features of the population growth, this algorithm can be useful for managing and exploiting populations. We are less interested here in control obtained by using $K<0$, which represents constant positive migration, although this approach has been discussed in the literature. For example, McCallum [21] and Stone [22] studied the effect of such positive migration on several one-dimensional maps (e.g., the exponential map) by providing numerical evidence of the rich dynamics that emerge.

Recently, Parthasarathy and Sinha [23,24] illustrated a method for suppressing chaos by means of a constant feedback as described in Eq. (2). They investigated numerically the logistic and the exponential maps, with some specific choice of parameters and initial conditions. Graphs of x_n versus n were plotted, showing some number of iterations of the map (2). These graphs demonstrated apparent decay to a fixed point and apparent periodic behavior with some values of feedback K, in contrast to apparent chaotic behavior without feedback (i.e., $K=0$). However, no proof of convergence was given, no discussion of appropriate initial conditions was mentioned, and no rigorous argument was offered to show how to choose K. The K-r parameter space of the two maps was divided into regions displaying different behavior, based on numerical tests with successive increments of K and r values. It was suggested that the sign of the values of K that are capable of suppressing chaos depends in general on $f(x)$. This speculation is inaccurate, and we show here that it is always possible to choose $K>0$.

Numerical investigation of a discrete system may give some indication of its dynamics. It is extremely useful in both complicated and simple cases, and often seems to be the only feasible approach. Nevertheless, the possibility of drawing general conclusions from numerical investigation in this context is limited. A finite number of iterations started with some specific initial condition can at best demonstrate cases where the algorithm apparently works. It cannot prove that a sequence of iterations converges or displays chaotic or periodic behavior. Furthermore, to make the suggested algorithm applicable, one must provide a systematic method for choosing K and computing the range of appropriate initial conditions from which convergence is guaranteed. This is done in the next section.

II. CONTROL BY HARVESTING AT A CONSTANT RATE

We prove the following result: chaos or any periodic dynamics in unimodal maps can be controlled by *harvesting* the population at a constant rate, and there exists an interval of feasible values of harvesting rates that can be used for this purpose. For each such feasible value $(K>0)$, there exists an interval of initial conditions x_0 for which x_n, defined by Eq. (2), converges to a limit that is independent of x_0.

The underlying idea on which the algorithm is based is very simple and intuitive. Harvesting the population shifts the graph of the unimodal map downwards. The unstable fixed point around which we have undesired oscillations becomes stable and we can compute the appropriate basin of attraction. The precise formulation is given in the following theorem.

Theorem 1: Let $f\colon I\to I$ (where $I\subseteq[0,\infty)$) be a $\mathbb{C}^2$ function satisfying the following conditions: 1. $f(0)=0$. 2. $f(x)$ has two fixed points: $x=0$ and $x=x_Q>0$, with $f'(0)>1$ and $f'(x_Q)<-1$. 3. $f(x)$ attains its maximum at the point $M(x_M,f(x_M))$ for $0<x_M<x_Q$. 4. $f''(x)<0$ for $x\in[0,B]$ for some $B>x_Q$. 5. $f(x)>x$ for $x<x_Q$ and $f(x)<x$ for $x>x_Q$. Then there exist positive numbers $0<a_1<a_2$ such that for any choice of $K\in(a_1,\ a_2)$ the sequence

$$x_{n+1}=f(x_n)-K,\quad n=0,1,2,... \tag{3}$$

converges to $x_R(K)$ for any choice of $x_0\in(x_L(K),x_R(K))$, where $x_L(K)<x_R(K)$ are the two solutions of the equation $x+K=f(x)$.

Proof: We first prove the claim using only information on the interval $[0,x_M]$. In this case we can relax all the conditions on $f(x)$ concerning $x>x_M$ including unimodality. We consider the equation

$$f'(x)=1. \tag{4}$$

From the conditions given on $f(x)$ it follows that Eq. (4) has exactly one solution in the interval $(0,x_M)$. We denote this solution by x_P and set $a_2=f(x_P)-x_P$. We also define $a_1=f(x_M)-x_M$. Since $f'(x_P)=1$ and we have $f''(x)<0$ on $(0,\ x_M)$ we conclude that $0<a_1<a_2$. Now, for any given $K\in(a_1,a_2)$ the equation

$$x=f(x)-K \tag{5}$$

has exactly the two solutions on the interval $x\in(0,x_M)$. We denote these solutions by $x_L=x_L(K)$ and $x_R=x_R(K)$, where $x_L<x_R<x_M$.

Suppose now that for some $k\geqslant 0$ we have $x_k\in(x_L,\ x_R)$. Since on the interval (x_L,x_R) we have $f(x)-K>x$, we may conclude that $x_{k+1}=f(x_k)-K>x_k$. On the other hand, since $f(x)$ is monotonically increasing on (x_L,x_R) we also have $x_{k+1}=f(x_k)-K<f(x_R)-K$. Consequently, for any $x_0\in(x_L,x_R)$ we have by induction that x_n is a monotonically increasing bounded sequence, and is therefore convergent.

Its limit is a solution of Eq. (5), and since it is clearly not x_L, we finally have $\lim_{n\to\infty} x_n = x_R$, which proves the desired result.

This construction does not use any information on the behavior of $f(x)$ beyond x_M. However, if we use such information on $f(x)$, we can extend the interval from which we are allowed to choose K. In this case, the convergence to the limit is not necessarity monotonic. We consider the equation

$$f'(x) = -1 \tag{6}$$

From the conditions given on $f(x)$ we conclude that Eq. (6) has a unique solution on the interval $x_M < x < x_Q$. We denote this solution by x_S and set $a_1' = f(x_M) - x_S > 0$. The conditions given on $f(x)$ assure that $0 < a_1' < a_1$. For any $K \in (a_1', a_1)$ we define $x_L(K)$ and $x_R(K)$ as the two solutions of Eq. (5), as above. To complete the proof we show that if $x_0 \in (x_L, x_R)$ and $K \in (a_1', a_1)$ the sequence x_n converges to x_R. The sequence x_n is bounded by $f(x_M) - K$. Since it cannot be monotonically increasing, it follows that there is an index k where $x_k \in (x_M, f(x_M) - K)$. Since the choice of K implies $f(x_M) - x_S < K < f(x_M) - x_M$, and $f''(x) < 0$ in this interval, we have $-1 = f'(x_S) < f'(f(x_M) - K) < f'(x_M) = 0$. Therefore, the map (3) is a contraction on the discussed interval, and the result follows.

A. Remarks

1. A natural question that arises following the proof of the theorem is: What happens if x_0 is not chosen in the interval $x_0 \in (x_L, x_R)$, as required by the theorem? This question is actually related to the fact that the modified map (3) does not map the interval I to itself. Therefore, if the initial conditions are not chosen within the interval (x_L, x_R), we can not guarantee convergence to equilibrium. If $0 < x_0 < x_L$, the sequence x_n decreases until it eventually becomes negative. At that point we stop the iterations and declare the extinction of the population. If $x_0 > x_R$ then the dynamics depend on $f(x)$. The sequence x_n starts decreasing and after a finite number of iterations it either falls within the interval (x_L, x_R) and then converges as explained above, or falls below x_L where extinction occurs.

2. We also point out theorem 1 provides only sufficient conditions for convergence. Values of K outside the prescribed interval can either succeed or fail in suppressing chaotic or periodic oscillations, and can also drive the population to extinction (compare to the results reported in [24]). In general, this depends on the map and on x_0, as demonstrated below. In terms of population management, this means that inappropriate values of K are ‘‘dangerous.’’

3. Control by harvesting is not necessarily successful with unimodal maps that do not satisfy the conditions of theorem 1, particularly the smoothness requirements. One easy example is the map $f(x) = 1/2x$ when $x \in [0,1/2]$ and $f(x) = 1 - 1/2x$ when $x \in (1/2,1]$.

B. Summary of the method

1. Compute (x_M, y_M), the maximum point of $f(x)$. 2. Solve $f'(x) = 1$ on $(0,x_M)$ to obtain x_P. 3. Solve $f'(x) = -1$ on (x_M, x_Q) to obtain x_S. 4. Set $a_2 = f(x_P) - x_P$, $a_1 = f(x_M) - x_M$, $a_1' = f(x_M) - x_S$. 5. For any choice of $K \in (a_1', a_2)$ the sequence x_n is convergent, and if $K \in (a_1, a_2)$ it converges monotonically. 6. For any appropriate value of K, the initial condition x_0 may be any chosen value in the interval (x_L, x_R) where $x_L < x_R < x_M$ are the two solutions of the equation $f(x) = x + K$. The limit of x_n is x_R.

C. Examples

1. We demonstrate our method with the logistic map $f(x) = rx(1-x)$: $[0,1] \to [0,1]$ with $3 < r \leqslant 4$. Theorem 1 provides all the information required, and the calculations are straightforward:

$$\begin{aligned} a_2 &= f(x_P) - x_P = \frac{1}{4r}(r^2 - 2r + 1), \\ a_1 &= f(x_M) - x_M = \frac{1}{4}r - \frac{1}{2}, \\ a_1' &= f(x_M) - x_S = \frac{1}{4r}(r^2 - 2r - 2), \\ x_L, x_R &= \frac{-1 + r \pm \sqrt{1 - 2r + r^2 - 4rK}}{2r}. \end{aligned} \tag{7}$$

We consider the choice $r = 3.8$, which was the case studied in [16], where it was shown that 150 iterations of x_n defined by Eq. (3) and started from $x_0 = 0.3$ ‘‘apparently converge’’ to a fixed point if $K = 0.3$, whereas this apparently does not occur for $K = 0.2$ (period-2 orbits) or for $K = 0$ (chaotic orbits). Similar results were reported in this context in [24].

For this choice of r, theorem 1 assures that for all $0.318\,425 < K < 0.515\,78$ the sequence x_n converges to the limit $x_R(K) = x_R$. This holds for all initial values within $(x_L(K), x_R(K))$, specified in Eq. (7). Further, if $0.45 < K < 0.515\,789$ the sequence x_n is also monotonically increasing.

As explained above, theorem 1 provides only sufficient but not necessary conditions. For example, note that the choice $K = 0.3$ with $x_0 = 0.3$ that was used in [23] belongs to a region where in general the algorithm may or may not succeed, depending on the specific map. In this specific case, convergence occurs because $x_0 = 0.3 \Rightarrow x_1 = 0.4980$ and the map (3) is already a contraction in this region.

2. We examine the exponential map $f(x) = xe^{r(1-x)}$: $[0,\infty) \to [0,\infty)$ with the choice $r = 2.8$ as in [23] where plots of 150 iterations of x_n defined by Eq. (3) and started from $x_0 = 0.3$ were displayed. They indicated apparent convergence to a fixed point if $K = -0.95$ whereas this did not occur for $K = -0.5$ (period-2 orbits) or for $K = 0$ (chaotic orbits).

To use theorem 1 we solve the equations $(1 - rx)e^{r(1-x)} = \pm 1$ and obtain x_P and x_S. Clearly, $x_M = 1/r$, and the remaining calculations are straightforward. For $r = 2.8$ we find that if $1.731\,169 < K < 1.803\,445$ the sequence x_n converges to the fixed point $x_R(K)$, for all initial conditions $(x_L(K), x_R(K))$. To compute $x_L(K)$ and $x_R(K)$ for a given value of K one only needs to solve the equation $xe^{r^{(1-x)}}$

$=x+K$. Note that the interval of feasible values of $K>0$ decreases as r is increased, as indicated by the K-r parameter space map in [23].

III. DISCUSSION

There exist simple examples of discrete maps where it is easy to observe numerically rich dynamics but difficult to prove rigorously that this indeed occurs. Numerical simulations are crucial in many such cases and are the only feasible approach in others, as seen frequently in the literature. However, the ability to make general predictions based on a finite number of iterations of discrete maps is rather limited. It is therefore important to distinguish between numerical evidence and rigorous results.

Here, we have shown how to assure convergence of an algorithm for suppressing chaotic or periodic oscillations, for a whole class of one-dimensional maps without computing numerically even one orbit. This convergence is guaranteed for a whole range of initial conditions. The last point is often neglected in numerical reports. The method proposed here requires only solving some one-dimensional equations. Once this is done, and K and x_0 are chosen appropriately, the algorithm does not depend on the dynamics of the map any more. To the extent that such maps indeed capture the essential features of the population growth, this algorithm can be useful for managing and exploiting populations.

Finally, we point out that it is not obvious how our algorithm would perform in the presence of stochastic noise in the population, as may be the case in real systems. Particularly, it is not clear whether extinction would prevail over convergence to the desired fixed point. It is also not clear how to generalize the algorithm to multidimensional maps. These points may be interesting directions to pursue.

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